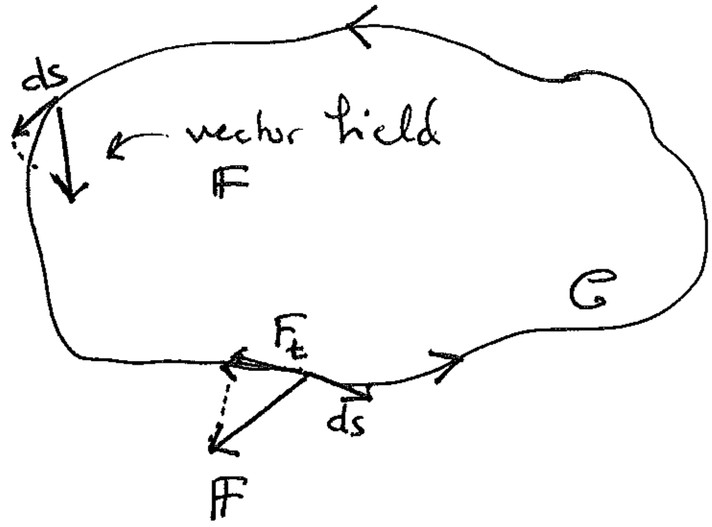
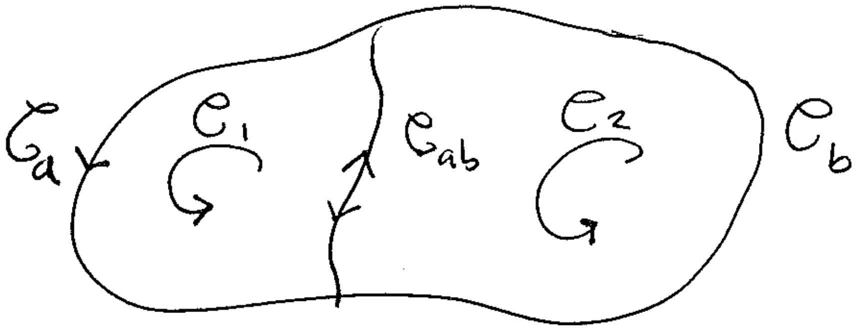
Physical intuition: circulation of a vector field.



Total circulation around $\mathcal{C}$

$$\oint_{\mathcal{C}} F_t \, ds = \oint_{\mathcal{C}} \mathbb{F} \cdot d\mathbf{r}$$

Now divide $\mathcal{C}$ into two loops:

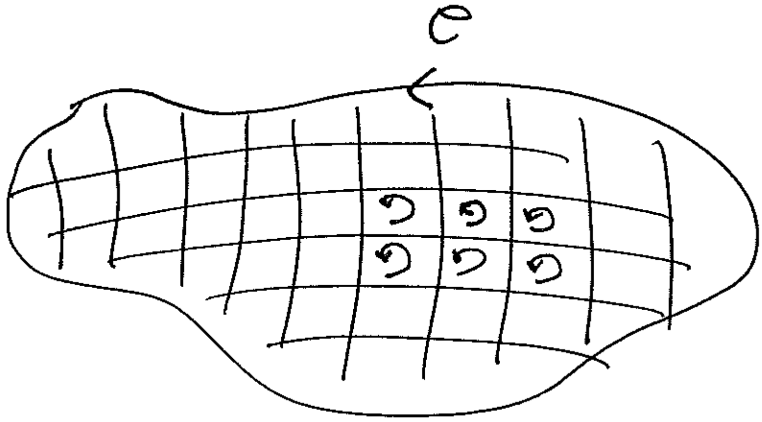


$$\mathcal{C}_1 = \mathcal{C}_a + \mathcal{C}_{ab} \qquad \mathcal{C}_2 = \mathcal{C}_b + \mathcal{C}_{ab}$$

$$\oint_{C} \mathbb{F}\cdot d\mathbb{r} = \left[\int_{C_a} + \int_{C_{ab}} - \int_{C_{ab}} + \int_{C_b}\right] \mathbb{F}\cdot d\mathbb{r}$$

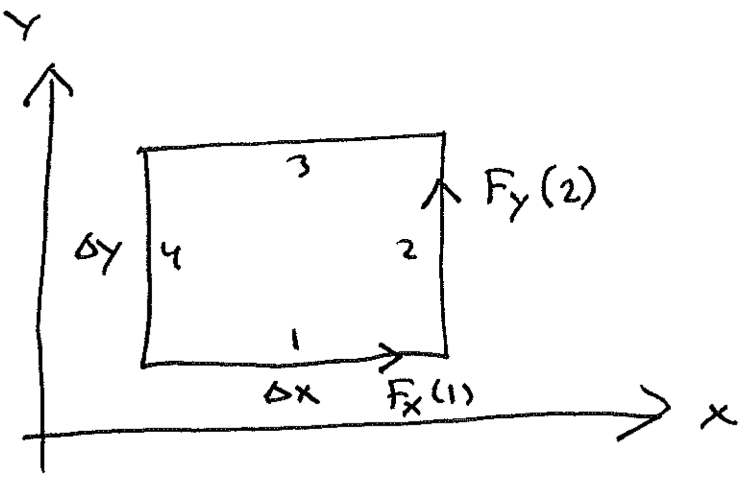
$$= \int_{C_a} \mathbb{F}\cdot d\mathbb{r} + \int_{C_b} \mathbb{F}\cdot d\mathbb{r}$$

The integration along the inner path cancels:
The same thing happens if we divide the region into small squares.



This holds no matter what shape the region has:

For each small square we can find the total circulation by summing the contributions from each side:



Total circulation: $F_x(1)\Delta x + F_y(2)\Delta y$

$$- F_x(3)\Delta x - F_y(4)\Delta y$$

How is $F_x(1)$ related to $F_x(3)$? View F_x as a function of y and Taylor expand to first order in Δy:

$$F_x(3) = F_x(1) + \left.\frac{\partial F_x}{\partial y}\right|_{(x,y)} \Delta y + O(\Delta y^2)$$

We can do a similar expansion for the other two terms

$$F_y(2) = F_y(4) + \frac{\partial F_y}{\partial x}\Delta x$$

The total circulation can then be written as

$$\big(F_x(1) - F_x(3)\big)\Delta x + \big(F_y(2) - F_y(4)\big)\Delta y$$

$$= \underbrace{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)}_{\text{This is precisely the z-component of curl}(\mathbb{F})\text{!}} \Delta x \Delta y$$

For many such squares we then obtain the formula for the circulation around any ~~region~~ surface S:

$$\oint_C \mathbb{F}\cdot d\mathbf{r} = \iint_S \mathrm{curl}(\mathbb{F})\cdot \mathbf{n}\, dS$$

This implies that we can view Stoke's theorem as a relation between the macroscopic and microscopic description of the circulation:

$$\underbrace{\oint_C \mathbb{F} \cdot d\mathbb{r}}_{\substack{\text{macroscopic circulat} \\ \text{around the surface} \\ S.}} = \underbrace{\iint_S \text{curl}(\mathbb{F}) \cdot d\mathbb{S}}_{\substack{\text{"sum" over all} \\ \text{microscopic circula} \\ \text{in infinitesimal squares} \\ \text{of area } dxdy.}}$$